Peridynamics and topology optimization

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WCSMO11-Sydney
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2. Sensitivity filtering in topology optimization
3. Equivalence on sensitivity filtering and topology optimization with peridynamic models
Peridynamics is a nonlocal model in Continuum Mechanics and Elasticity introduced by Silling \(^1\).

\(^1\)S. A. Silling, \textit{Reformulation of elasticity theory for discontinuities and long range forces}, JMPS, 48 (2000)
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- Nonlocality: points at a finite distance exert a force upon each other

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- Effective modeling for discontinuities in solids: cracks, fracture, cavitation,....

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Local elasticity models

A deformation $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is obtained as a minimizer of the hyperelastic (potential) energy

$$\int_\Omega W(Du(x)) \, dx + \int_\Omega f(x)u(x) \, dx,$$

where $\Omega$ is the reference solid, and $Du$ stands for the gradient of the deformation $u$. $f$ stands for external forces on the solid and $u$ verifies boundary conditions.
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- In the linear elastic case function $W$ is quadratic, so that the critical points equations is the linear elasticity system.
Local elasticity models

- A deformation $\mathbf{u} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is obtained as a minimizer of the hyperelastic (potential) energy

$$\int_{\Omega} W(D\mathbf{u}(\mathbf{x})) \, d\mathbf{x} + \int_{\Omega} f(\mathbf{x})\mathbf{u}(\mathbf{x}) \, d\mathbf{x},$$

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- In the linear elastic case function $W$ is quadratic, so that the critical points equations is the linear elasticity system.

- Existence of optimal solutions is mathematically guaranteed if the integrand $W$ is polyconvex (a convex function of $D\mathbf{u}$ and all its minors).
Nonlocal peridynamics models

- A deformation \( u : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is obtained as a minimizer of the macroelastic energy

\[
\int_{\Omega} \int_{\Omega \cap \{|x-x'|<\delta\}} w(x-x',u(x)-u(x')) \, dx \, dx' + \int_{\Omega} f(x)u(x) \, dx,
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where \( w \) is the pairwise potential function, and \( \delta \) is the horizon of interaction between particles.
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where $w$ is the pairwise potential function, and $\delta$ is the horizon of interaction between particles.

- This is a non local energy.

- In the linear case this is a quadratic functional,

$$
\int_{\Omega} \int_{\Omega} K(x, x') (\mathbf{u}(x) - \mathbf{u}(x'))^2 \, dx \, dx',
$$

where $K$ is a convolution kernel.
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Nonlocal peridynamics models

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- It has shown to be a very good way for modeling fracture and discontinuities
- There are already numerical algorithms available for simulation
Nonlocal peridynamics models

- Existence of minimizers for this problems is guaranteed by a non-local convexity notion

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Nonlocal peridynamics models

- Existence of minimizers for this problems is guaranteed by a non-local convexity notion: we say that $w$ verifies the non-local convexity property if the function

$$y \to \int_{\Omega} w(x - x', y - u(x')) \, dx'$$

is convex for any point $x$ and any test function $u$.


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- It can be shown that when the horizon $\delta \to 0$, nonlocal peridynamics models converge to a local hyperelastic model.

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Sensitivity filtering is a fundamental tool in topology optimization in order to avoid mesh dependence and checkerboard problem.
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If $c$ is the cost functional (compliance) and $\rho$ the density variable (optimization variable), sensitivity filter modifies in the numerical algorithm $\frac{\partial c}{\partial \rho}$ by a filter sensitivity $\hat{\frac{\partial c}{\partial \rho}}$ that can be obtained as convolution of the previous one,

$$\hat{\frac{\partial c}{\partial \rho}}(x) = \int K(x, y) \frac{\partial c}{\partial \rho}(y) \, dy$$
Sensitivity filtering

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A convolution filtered function can be obtained as the solution of a Helmholtz-type PDE equation. Applying a sensitivity filter in compliance optimization is equivalent to optimized a nonlocal elasticity compliance: we replace the linear elasticity state by a nonlocal one in the compliance optimization problem. The nonlocal model consists of two coupled PDE:

- The linear elasticity system;

References:

Sensitivity filtering

- A convolution filtered function can be obtained as the solution of a Helmholtz-type PDE equation \(^4\).

- Applying a sensitivity filter in compliance optimization is equivalent to optimized a nonlocal elasticity compliance: we replace the linear elasticity state by a nonlocal one in the compliance optimization problem \(^5\). The nonlocal model consists of two coupled PDE:
  - The linear elasticity system;
  - A Helmholtz filtering equation for displacement filtering (gradient enhancement-nonlocality)

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Question

Could the sensitivity filter be obtained as the derivative of the nonlocal peridynamic compliance?
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Problem (WORK IN PROGRESS)

Filtered sensitivities are peridynamic compliance sensitivities
IDEA OF THE PROOF

Naturally we focus on the linear case, in which the peridynamics functional is quadratic

\[ I(u) = \int_{\Omega} \int_{\Omega} K(x, x')(u(x) - u(x'))^2 \, dx \, dx' \]

where \( K \) is convolution kernel.
Let $u$ be a minimizer (displacement).
Sensitivity filtering and peridynamics

Let \( u \) be a minimizer (displacement). It is elementary to check that

\[
\frac{dl}{dt}(u + tv)_{t=0} = 4 \int_{\Omega} v(x) \int_{\Omega} K(x, x')(u(x) - u(x')) \, dx \, dx'
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Therefore the (FRECHET) derivate of the nonlocal functional is

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Sensitivity filtering and peridynamics

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Therefore the (FRECHET) derivate of the nonlocal functional is

$$l'(u) = \int_{\Omega} K(x, x')(u(x) - u(x')) \, dx \, dx'$$

Then, if $u$ is an optimal solution of the peridynamics functional (i.e. a deformation) then it can be written as **convolution**

$$u(x) = C \int_{\Omega} K(x, x')u(x') \, dx'$$

therefore, the **peridynamics displacement can be written as the solution of a Helmholtz-type PDE**
THANK YOU FOR PAYING ATTENTION