EXPLICIT QUASICONVEXIFICATION FOR SOME COST FUNCTIONALS DEPENDING ON DERIVATIVES OF THE STATE IN OPTIMAL DESIGN

JOSÉ C. BELLIDO AND PABLO PEDREGAL
Departamento de Matemáticas
Universidad de Castilla-La Mancha
c/ Campus Universitario s.n., 13.071-Ciudad Real, Spain
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Abstract. We study relaxation for optimal design problems in conductivity in the two-dimensional situation. To this end, we reformulate the optimal design problem in an equivalent way as a genuine vector variational problem, and then analyze relaxation of this new variational problem. Our main achievement is to explicitly compute the quasiconvexification of the involved density in this problem for some interesting cases. We think the method given here could be generalized to compute quasiconvex envelopes in other situations. We restrict attention to the two-dimensional case.

1. Introduction. We would like to consider in this paper a typical optimal design problem in conductivity. We have to determine how to mix two conducting materials, with conductivities, or dielectric permittivities, \(\alpha\) and \(\beta\), \(0 < \alpha < \beta\), to fill out a domain \(\Omega \subset \mathbb{R}^2\) in such a way that we minimize the cost functional

\[
I(\chi) = \int_{\Omega} \left[ \varphi(x, \alpha \chi(x) + \beta(1 - \chi(x)), u(x), \nabla u(x)) \right] dx,
\]

where the design variable, \(\chi\), is the characteristic function of a subset of \(\Omega\) occupied by the material with conductivity \(\alpha\), and \(u \in H^1(\Omega)\) is the unique solution of the state equation

\[
\begin{aligned}
- \text{div}((\alpha \chi(x) + \beta(1 - \chi(x))\nabla u(x)) = P(x), & \quad \text{in } \Omega, \\
u = u_0, & \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(u_0 \in H^1(\Omega)\) and \(P \in H^{-1}(\Omega)\) stand for the exterior charges performing in \(\Omega\).

\[\varphi : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^* = \mathbb{R} \cup \{+\infty\}\]

is supposed to be a Carathéodory function.

If we call

\[\varphi_\alpha(x, u, \nabla u) = \varphi(x, \alpha, u, \nabla u)\]

and

\[\varphi_\beta(x, u, \nabla u) = \varphi(x, \beta, u, \nabla u),\]

the cost functional can be rewritten in the following way

\[
I(\chi) = \int_{\Omega} \left[ \chi(x)\varphi_\alpha(x, u(x), \nabla u(x)) + (1 - \chi(x))\varphi_\beta(x, u(x), \nabla u(x)) \right] dx.
\]

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General optimal design problems of this kind fail to have optimal solutions in the class of characteristic functions (see [15]). When the cost functional does not depend on the derivatives of the state or it does in a suitable way, the main tool to analyze and understand this lack of existence has been Homogenization Theory. See, for instance, [5, 16, 17, 18]. The recent survey [24] is an excellent introduction to the homogenization method in optimal design. In this approach the relaxation of the optimal design problem is obtained enlarging, in a proper way, the set of admissible designs until the “G-closure” of the initial admissible set. The characterization of such a G-closure, or at least of the extremal points of this set, is essential in this problem. See [11, 12, 22]. When the cost does depend on the derivatives of the state, new ideas have been developed to deal with this kind of problems for the particular case of a quadratic functional on the gradient of the state ([8, 10, 23]). Lately, an alternative approach has been proposed in [3, 4, 20], consisting in avoiding the nonlocal nature of the state equation by reformulating the optimal design problem as a genuine vector variational problem, and then studying this new problem with the usual techniques of the Calculus of Variations. Our main goal here is to explicitly compute the relaxed density for this variational problem in some interesting cases of objective functions. This paper was announced in [21] where that relaxed density was computed for the case in which the cost functional is

$$I(\chi) = \int_\Omega |\nabla u(x) - F(x)|^2 \, dx,$$

where $F$ stands for a target field and the cost measures the mean-square deviation of the underlying electric field from it. Here we explore how far the computations in [21] can be pushed.

Let us remind readers how we reformulate the optimal design problem as a variational problem. Let $p \in H_0^1(\Omega)$ be the solution of the Poisson’s equation

$$-\Delta p = P, \quad \text{in } H^{-1}(\Omega).$$

Then our equilibrium equation can be rewritten as

$$-\text{div}(\alpha \chi(x) + \beta (1 - \chi(x)) \nabla u(x) - \nabla p(x)) = 0, \quad \text{in } \Omega.$$

Under the hypothesis of simple connectedness of $\Omega$, there exists a stream function $v \in H^1(\Omega)$ such that the previous equation is equivalent to

$$(\alpha \chi(x) + \beta (1 - \chi(x)) \nabla u(x) + T \nabla v(x) = \nabla p(x),$$

where $T$ is the counterclockwise $\frac{\pi}{2}$-rotation in the plane (see [7]). If we put both $u$ and $v$ in a single vector field $U = (u, v)$ then it is not hard to realize that our initial optimal design problem is equivalent to

$$\text{Minimize } I(U) = \int_\Omega W(x, U(x), \nabla U(x)) \, dx,$$

subject to

$$U \in H^1(\Omega; \mathbb{R}^2), \quad U^{(1)} = u_0 \quad \text{on } \partial \Omega,$$

where the density

$$W : \Omega \times \mathbb{R}^2 \times \mathbb{M}^{2 \times 2} \to \mathbb{R}^+$$

is given by

$$W(x, U, A) = \begin{cases} \phi_\alpha(x, U^{(1)}, A^{(1)}), & \text{if } A \in \Lambda_{\alpha,x} \backslash \Lambda_{\beta,x}, \\ \phi_\beta(x, U^{(1)}, A^{(1)}), & \text{if } A \in \Lambda_{\beta,x} \backslash \Lambda_{\alpha,x}, \\ \min\{\phi_\alpha(x, U^{(1)}, A^{(1)}), \phi_\beta(x, U^{(1)}, A^{(1)})\}, & \text{if } A \in \Lambda_{\alpha,x} \cap \Lambda_{\beta,x}, \\ +\infty, & \text{otherwise.} \end{cases}$$
Here $U^{(i)}$, $i = 1, 2$, denotes the $i$-th component of $U$, $A^{(i)}$, $i = 1, 2$, denotes the $i$-th row of $A$, and $\Lambda_{\gamma,x}$ is the two-dimensional affine manifold
\[
\Lambda_{\gamma,x} = \left\{ A \in \mathbb{M}^{2 \times 2} : \gamma A^{(1)} + TA^{(2)} = \nabla p(x) \right\} = V^x + \Lambda_{\gamma},
\]
where
\[
\Lambda_{\gamma} = \left\{ A \in \mathbb{M}^{2 \times 2} : \gamma A^{(1)} + TA^{(2)} = 0 \right\}
\]
and
\[
V^x = \left( \begin{array}{c} 0 \\ -T \nabla p(x) \end{array} \right).
\]
Observe that $\Lambda_{\alpha,x} \cap \Lambda_{\beta,x} = \{ V^x \}$. The equivalence of this new variational problem with our original optimal design problem was rigorously established in [3] for a more general class of optimal design problems in the one-dimensional situation, the extension to the two-dimensional case is straightforward.

The above perspective based on the introduction of potentials to avoid differential constraints has been successfully used in other contexts [9, 14, 25].

Our main achievement is to provide a complete analysis of relaxation for such optimal design problem. In particular, we give the fully explicit expression of the quasiconvexification of $W$, $QW$, in some interesting cases. To state the main result we need to introduce a little bit of notation. Put
\[
g(B) = (\alpha + \beta)^2 \left[ (\alpha B^{(1)} + TB^{(2)}) \cdot (\beta B^{(1)} + TB^{(2)}) \right]^2
- 4\alpha \beta \left[ \alpha B^{(1)} + TB^{(2)} \right] \left[ \beta B^{(1)} + TB^{(2)} \right]^2,
\]
\[
h(B) = \left( \alpha B^{(1)} + TB^{(2)} \right) \cdot \left( \beta B^{(1)} + TB^{(2)} \right).
\]
Let $r_i(B)$, $i = 1, 2$, be
\[
r_1(B) = \frac{1}{2} + \frac{1}{2(\beta - \alpha) \det B} \left[ \alpha \beta \left| B^{(1)} \right|^2 - \left| B^{(2)} \right|^2 - \sqrt{g(B)} \right],
\]
\[
r_2(B) = \frac{1}{2} + \frac{1}{2(\beta - \alpha) \det B} \left[ \alpha \beta \left| B^{(1)} \right|^2 - \left| B^{(2)} \right|^2 + \sqrt{g(B)} \right].
\]
The vectors $B_i$, $i = 1, 2$, are
\[
B_1 = \frac{1}{\beta - \alpha} \left( \beta B^{(1)} + TB^{(2)} \right), \quad B_2 = \frac{-1}{\beta - \alpha} \left( \alpha B^{(1)} + TB^{(2)} \right).
\] Finally $\Gamma$ is the set of matrices
\[
\Gamma = \left\{ B \in \mathbb{R}^{2 \times 2} : g(B) \geq 0, h(B) \leq 0 \right\}.
\]

**Theorem 1.1.** Assume that, for fixed $(x, u)$,
1. the functions $\varphi_{\alpha}(x, u, \nabla u)$, $\varphi_{\beta}(x, u, \nabla u)$ are convex on $\nabla u$ and non-negative;
2. for all $A \in \mathbb{M}^{2 \times 2}$ such that $A^x = A - V^x \in \Gamma$, the functions $g_A(t) = t \varphi_{\alpha} \left( x, u, \frac{1}{t} A^x_1 \right) + (1 - t) \varphi_{\beta} \left( x, u, \frac{1}{1 - t} A^x_2 \right)$, $t \in (0, 1)$, where $A^x_1$ and $A^x_2$ are given by (1.1), are such that
\[
\min_{t \in [r_1(A^x), r_2(A^x)]} g_A(t) = \min_{t \in [r_1(A^x), r_2(A^x)]} g_A(t).
\]
Then the quasiconvexification of $W$, $QW(x, U, A)$, is given by

$$
\min_{i=1,2} \left\{ \frac{1}{r_i(A^x)} \varphi_\alpha \left( x, U^{(1)}, \frac{1}{r_i(A^x)} A_1^x \right) + (1 - r_i(A^x)) \varphi_\beta \left( x, U^{(1)}, \frac{1}{1 - r_i(A^x)} A_2^x \right) \right\},
$$

if $A^x \in \Gamma$. If $A^x \not\in \Gamma$ then $QW(x, U, A) = +\infty$. Moreover there is an optimal microstructure in the form of a first-order laminate. In addition, if either $\varphi_\alpha$ and $\varphi_\beta$ are strictly convex, or the minimum in (1.2) is only attained at the end-points of $[r_1(A^x), r_2(A^x)]$, then the unique optimal microstructures are first order laminates.

This theorem applies in many situations for which we can explicitly compute the relaxed density. For instance, let us consider the objective function

$$
\varphi(\nabla u(x)) = |\nabla u(x)|^p,
$$

where $p \geq 0$. Then the density in the equivalent variational problem is

$$
W(\nabla U(x)) = \left| \nabla U^{(1)}(x) \right|^p,
$$

if $\nabla U(x) \in \Lambda_{\alpha,x} \cup \Lambda_{\beta,x}$, and

$$
W(\nabla U(x)) = +\infty,
$$

otherwise. If $p = 1$, we obtain

$$
QW(A) = (|A_1^x| + |A_2^x|),
$$

where $A_1^x$ and $A_2^x$ are given in (1.1), if $A^x \in \Gamma$, and

$$
QW(A) = +\infty,
$$

otherwise. In the case $p = 2$, we obtain the more complicated formula

$$
QW(A) = \frac{1}{2\alpha\beta} \left[ \alpha\beta |A^{(1)}|^2 - |A^{(2)}|^2 + (\alpha + \beta) \det A^x - \sqrt{g(A^x)} \right],
$$

if $A^x \in \Gamma$, and

$$
QW(A) = +\infty,
$$

otherwise.

The next section is devoted to proving Theorem 1, while several examples and remarks will be commented in Section 4. In Section 3 we discuss relaxation of our equivalent variational problem. Indeed, this is a delicate issue due to the fact that the integrand $W$ is not a Carathéodory function, because it takes on the value $+\infty$ suddenly, and the classical results of relaxation in the Calculus of Variations cannot be applied directly. We will however establish the relaxation result we need in our situation. In fact, if we assume the typical bounds

$$
e \leq \varphi_\alpha(x, u, \lambda), \varphi_\beta(x, u, \lambda) \leq h_1(x, u) + h_2(x, u)|\lambda|^2,
$$

(1.3)

where $c$ is a constant and $h_1, h_2$ are locally bounded functions and $h_2 \geq 0$, we can prove the two following theorems. Notice that no coercivity is assumed on $\varphi_\alpha$ and $\varphi_\beta$. The first one is a relaxation result in terms of gradient Young measures and the second one is a relaxation result involving the appropriate convex envelope of $W$.

**Theorem 1.2.** In addition to (1.3), assume (for simplicity) that

$$
\varphi_\alpha(x, u, 0) = \varphi_\beta(x, u, 0),
$$

a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^2$. The infima

$$
m = \inf \left\{ \int_{\Omega} W(x, U(x), \nabla U(x)) \, dx : U \in H^1(\Omega; \mathbb{R}^2), U^{(1)} - u_0 \in H^1_0(\Omega) \right\}
$$
and
\[
\bar{m} = \inf \left\{ \int_{\Omega} \int_{M^{2 \times 2}} W(x, U(x), A) \, d\nu_x(A) \, dx : \nu_x \in \mathcal{A}, \, \nabla U(x) = \int_{M^{2 \times 2}} A \, d\nu_x(A) \right\},
\]
where \( \mathcal{A} \) is the set of homogeneous \( H^1 \)-Young measures, coincide. Moreover, for any gradient Young measure \( \nu = \{\nu_x\}_{x \in \Omega} \) with
\[
\nabla U(x) = \int_{M^{2 \times 2}} A \, d\nu_x(A), \quad U \in H^1(\Omega; \mathbb{R}^2), \quad U^{(1)} - u_0 \in H^1_0(\Omega),
\]
and \( \text{supp}(\nu_x) \subset \Lambda_x = \Lambda_{\alpha,x} \cup \Lambda_{\beta,x} \), there exists a bounded sequence of gradients \( \{\nabla U_j\} \) generating \( \nu \) such that
\[
\nabla U_j(x) \in \Lambda_x, \quad \text{for all } j, \text{ a.e. } x \in \Omega,
\]
and
\[
\lim_{j \to \infty} \int_{\Omega} W(x, U_j(x), \nabla U_j(x)) \, dx = \int_{\Omega} \int_{M^{2 \times 2}} W(x, U(x), A) \, d\nu_x(A) \, dx.
\]

The equality of the infima \( m \) and \( \bar{m} \) is true in general when the integrand is a Carathéodory function. However, when it is not, the definition of the quasiconvex envelope may be different when taken as an infimum over gradients ([6]) or in terms of gradient Young measures ([19]). Indeed, based on the preceding result all these questions can be clarified in our situation, because we can show that the quasiconvexification is well-defined by proving that the definitions in terms of gradients or in terms of gradient Young measures coincide and the envelope so defined is a quasiconvex function. These questions will be precisely analyze in Section 3.

We insist in the fact that we do not need any coercivity bound on \( \varphi_\alpha \) and \( \varphi_\beta \) to establish the preceding result. This is a consequence of the structure of the set where \( W \) is finite.

To state the next theorem, let us consider the infimum \( \hat{m} \)
\[
\inf \left\{ \hat{I}(U) = \int_{\Omega} QW(x, U(x), \nabla U(x)) \, dx : U \in H^1(\Omega; \mathbb{R}^2), \, U^{(1)} - u_0 \in H^1_0(\Omega) \right\}.
\]

**Theorem 1.3.** The equalities
\[
m = \bar{m} = \hat{m},
\]
hold, and the infima \( \hat{m} \) and \( \bar{m} \) are attained.

As before, this result is true in general for Carathéodory integrands. The equality \( m = \hat{m} \) is a classical result ([6]) in that case, but some extra work is required when the integrand fails to be a Carathéodory function.
2. Proof of Theorem 1. This section is devoted to proving Theorem 1. For a fixed pair \((x, U) \in \Omega \times \mathbb{R}^2\), we have seen that in computing the quasiconvexification \(QW(x, U, A)\) we must care about homogeneous gradient Young measures supported in the set where the function \(W(x, U, \cdot)\) is finite, \(\Lambda_x\), and with first moment \(A\). Thus we examine the minimization problem

\[
\text{Minimize } \int_{\mathbb{M}^{2 \times 2}} W(x, U, F) \, d\nu(F),
\]

where \(\nu\) runs through all the gradients Young measures supported in \(\Lambda_x\) with given first moment \(A\). We will prove the theorem for the simplest case of a function \(W\) not depending on \((x, U)\). This means that in the optimal design problem \(\varphi_\alpha\) and \(\varphi_\beta\) do not depend on \((x, u)\), and \(P \equiv 0\). So that, we are concerned about

\[
QW(A) = \min \left\{ \int_{\mathbb{M}^{2 \times 2}} W(F) \, d\nu(F) : \nu \in \mathcal{A}, \supp \nu \subset \Lambda, A = \int_{\mathbb{M}^{2 \times 2}} F \, d\nu(F) \right\},
\]

(2.5)

where \(\Lambda = \Lambda_\alpha \cup \Lambda_\beta\). The generalization to obtain the value of the minimum (2.4) and its optimal microstructures (gradient Young measures), as it is stated in Theorem 2, can be carried out very easily. It amounts to deal with translations.

For the proof, we first find a lower bound of the quasiconvexification and then we prove the bound is optimal finding microstructures attaining the bound. The optimal microstructures we find are first-order laminates. We proceed in several steps, following closely [21].

**Step 1.** Let \(\nu\) be a homogeneous gradient Young measure supported in \(\Lambda = \Lambda_\alpha \cup \Lambda_\beta\) with first moment \(A\). We can decompose \(\nu\) as

\[
\nu = t\nu_\alpha + (1 - t)\nu_\beta, \quad t \in [0, 1],
\]

where \(\nu_\alpha\) and \(\nu_\beta\) are probability measure such that

\[
\text{supp}(\nu_\alpha) \subset \Lambda_\alpha, \quad \text{supp}(\nu_\beta) \subset \Lambda_\beta.
\]

Let \(A_\alpha\) and \(A_\beta\) be the matrices

\[
A_\alpha = \int_{\Lambda_\alpha} F \, d\nu_\alpha(F) \in \Lambda_\alpha, \quad A_\beta = \int_{\Lambda_\beta} F \, d\nu_\beta(F) \in \Lambda_\beta.
\]

For a matrix \(A \in \mathbb{M}^{2 \times 2}\) it is very easy to check that

\[
\det A = -A^{(1)} \cdot TA^{(2)},
\]

\[
\det A = \alpha \left| A^{(1)} \right|^2 \quad \text{if } A \in \Lambda_\alpha,
\]

\[
\det A = \beta \left| A^{(1)} \right|^2 \quad \text{if } A \in \Lambda_\beta,
\]

and

\[
\det(tA_\alpha + (1 - t)A_\beta) = t \det A_\alpha + (1 - t) \det A_\beta - t(1 - t) \det(A_\alpha - A_\beta).
\]

This last formula is only valid for \(2 \times 2\) matrices.

We will use the equality

\[
\int_{\mathbb{M}^{2 \times 2}} \det F \, d\nu(F) = \det \left( \int_{\mathbb{M}^{2 \times 2}} F \, d\nu(F) \right).
\]

(2.6)

This is the typical consequence of the weak lower semicontinuity of the determinant. However, this equality is not true for any homogeneous \(H^1\)-Young measure since we need a little bit more of integrability for the generating sequence of the Young
measure (see [13, 19]). Notice that the measures we are taking into account are peculiar in the sense that they are supported in $\Lambda$, and this suffices, as a consequence of Theorem 2, to prove that (2.6) is true in this case. Indeed, if $\nu$ is a homogeneous $H^1$-Young measure supported in $\Lambda$, by Theorem 2, we know there exists a sequence of gradients $\{\nabla U_j\}$ generating $\nu$, such that $\nabla U_j(x) \in \Lambda$ a.e. $x \in \Omega$ and for all $j$, and $\{\nabla U_j^2\}$ is equi-integrable. As a consequence the sequence $\{\det \nabla U_j\}$ is weakly convergent in $L^1(\Omega)$.

By using (2.6) and all the decompositions written above, it is elementary to arrive at

$$\alpha t \int_{\Lambda_\alpha} |F^{(1)}|^2 d\nu_\alpha(F) + \beta(1-t) \int_{\Lambda_\beta} |F^{(1)}|^2 d\nu_\beta(F) =$$

$$\alpha t \left| \int_{\Lambda_\alpha} F^{(1)} d\nu_\alpha(F) \right|^2 + \beta(1-t) \left| \int_{\Lambda_\beta} F^{(1)} d\nu_\beta(F) \right|^2 - t(1-t) \det(A_\alpha - A_\beta).$$

By Jensen’s inequality we conclude

$$\det(A_\alpha - A_\beta) \leq 0.$$

**Step 2. A lower bound for the quasiconvexification.** The objective functional of our minimization problem (2.5) can be rewritten, using the above decomposition for our feasible probability measures, as

$$t \int_{\Lambda_\alpha} \varphi_\alpha \left( F^{(1)} \right) d\nu_\alpha(F) + (1-t) \int_{\Lambda_\beta} \varphi_\beta \left( F^{(1)} \right) d\nu_\beta(F),$$

and, once again by using Jensen’s inequality, a lower bound for this expression is

$$t \varphi_\alpha \left( A^{(1)}_\alpha \right) + (1-t) \varphi_\beta \left( A^{(1)}_\beta \right).$$

We consider the following finite-dimensional optimization problem

$$\text{Minimize} \quad t \varphi_\alpha \left( A^{(1)}_\alpha \right) + (1-t) \varphi_\beta \left( A^{(1)}_\beta \right) \quad \text{(2.7)}$$

subject to

$$A = tA_\alpha + (1-t)A_\beta, \quad A_\alpha \in \Lambda_\alpha, \ A_\beta \in \Lambda_\beta, \ t \in [0,1],$$

$$\det(A_\alpha - A_\beta) \leq 0.$$
We notice that $A_1, A_2$ cannot vanish unless $A \in \Lambda$. Then we can rewrite the above optimization problem, (2.7), in the following terms

\[
\text{Minimize } t \phi_\alpha \left( \frac{1}{t} A_1 \right) + (1 - t) \phi_\beta \left( \frac{1}{1 - t} A_2 \right)
\]

subject to

\[
t \in (0, 1), \quad \left| A_1 \right|^2 \frac{\alpha}{t^2} + \left| A_2 \right|^2 \frac{\beta}{(1 - t)^2} - A_1 \cdot A_2 \frac{\alpha + \beta}{t(1 - t)} \leq 0.
\]

We notice that this last expression is precisely $\det(A_\alpha - A_\beta)$, and the vectors $A_1$ and $A_2$ are constant depending on $A$.

Put

\[
\psi_A(t) = \left| A_1 \right|^2 \frac{\alpha}{t^2} + \left| A_2 \right|^2 \frac{\beta}{(1 - t)^2} - A_1 \cdot A_2 \frac{\alpha + \beta}{t(1 - t)}.
\]

For fixed $A$, let us see what the values $t \in [0, 1]$ for which $\psi_A(t) \leq 0$ are. To this end, we consider the quadratic equation in $t$, $\psi_A(t) = 0$. Indeed, it can be rewritten as

\[
\alpha (1 - t)^2 \left| A_1 \right|^2 + \beta t^2 \left| A_2 \right|^2 - (\alpha - \beta) t(1 - t) A_1 \cdot A_2 = 0.
\]

The value of this parabola for $t = 0$ and $t = 1$ is positive if $A$ does not belong to either $\Lambda_\alpha$ or $\Lambda_\beta$. Hence, the two roots of this equation, if they exist, are in $[0, 1]$ or are outside. On the other hand, this equation has solutions if the discriminant is positive, and the solutions are in $(0, 1)$ if the leading coefficient is strictly positive and the vertex belongs to $(0, 1)$. After a few computations we have that the quadratic equation is

\[
\det A t^2 - \frac{1}{\beta - \alpha} \left( \alpha \beta \left| A^{(1)} \right|^2 - \left| A^{(2)} \right|^2 + (\beta - \alpha) \det A \right) t
\]

\[
+ \frac{1}{(\beta - \alpha)^2} \left( \alpha \beta^2 \left| A^{(1)} \right|^2 + \alpha \left| A^{(2)} \right|^2 - 2 \alpha \beta \det A \right) = 0.
\]

It is elementary to check that the three conditions mentioned above are

\[
g(A) \geq 0, \quad h(A) \leq 0.
\]

Therefore, if the matrix $A$ does not satisfy those conditions the minimum in (2.7) will be $+\infty$, and consequently also the quasiconvexification $QW(A) = +\infty$.

Let $A \in \Gamma$. The set of admissible $t$'s for our optimization problem is the subinterval of $(0, 1)$, $[r_1(A), r_2(A)]$, where $r_i(A)$, $i = 1, 2$, are the two roots of $\psi_A(t) = 0$. It is very easy to check that $r_i(A)$, $i = 1, 2$, are given by the formulas before the statement of Theorem 1.

By all the above remarks is clear that for $A \in \Gamma$ the minimization problem (2.7) can be rewritten as

\[
\text{Minimize } t \phi_\alpha \left( \frac{1}{t} A_1 \right) + (1 - t) \phi_\beta \left( \frac{1}{1 - t} A_2 \right)
\]

subject to

\[
t \in [r_1(A), r_2(A)],
\]

and by hypothesis we know this minimum is attained in $\{r_1(A), r_2(A)\}$.

Step 3. The lower bound is optimal. We have seen in the preceding step that

\[
QW(A) \geq \min_{i=1,2} \left\{ r_i(A) \phi_\alpha \left( \frac{1}{r_i(A)} A_1 \right) + (1 - r_i(A)) \phi_\beta \left( \frac{1}{1 - r_i(A)} A_2 \right) \right\},
\]

if $A \in \Lambda$, and

\[
QW(A) = +\infty,
\]
otherwise. Now we prove that, in fact, the equality holds. This is so, because there exists a first-order laminate for which we have the equality, that is to say, they are optimal microstructures. Clearly, for fixed $A$ this laminate is one of the two laminates

$$r_i(A)\delta_{A,\alpha,i} + (1 - r_i(A))\delta_{A,\beta,i},$$

with

$$A_{\alpha,i} = \left( \begin{array}{c} z_i \\ \alpha T z_i \end{array} \right), \quad z_i = \frac{1}{r_i(A)(\beta - \alpha)} \left( \beta A^{(1)} + TA^{(2)} \right)$$

and

$$A_{\beta,i} = \left( \begin{array}{c} w_i \\ \beta T w_i \end{array} \right), \quad w_i = \frac{-1}{(1 - r_i(A))(\beta - \alpha)} \left( \alpha A^{(1)} + TA^{(2)} \right),$$

$i = 1, 2$. It is very easy to check that $\det (A_{\beta,i} - A_{\alpha,i}) = 0$, $i = 1, 2$.

The second part of the theorem is straightforward following the above remarks. This finishes the proof.

3. Relaxation. In this section we prove the relaxation Theorems 2 and 3. We will also show that the quasiconvexification of $W$ is well-defined in spite of $W$ not being a Carathéodory function. We will need the following lemma in the sequel. It is important in itself because it claims that any sequence of gradients satisfying a certain pointwise bound it is bounded in $L^2$. In particular this result yields the coercivity of the functional $I$ as we will see in the proof of Theorem 2.

Lemma 3.1. Let $\{U_j\}$ be a sequence in $H^1(\Omega; \mathbb{R}^2)$ such that

$$U_j^{(1)} = u_0, \quad \text{on } \partial \Omega,$$

$$|\nabla U_j(x)|^2 \leq s \det \left( \nabla U_j(x) \right),$$

a.e. $x \in \Omega$ and for all $j$ where $s > 0$ is a constant and $u_0 \in H^1(\Omega)$. Then there exists a new constant $C > 0$, independent of the sequence, such that

$$\|\nabla U_j\|_{L^2(\Omega; \mathbb{M}^{2 \times 2})} \leq C,$$

for all $j$.

Proof. Integrating the inequality in the hypothesis, we obtain

$$\int_\Omega |\nabla U_j(x)|^2 \, dx \leq s \int_\Omega \det (\nabla U_j(x)) \, dx.$$

We assert that the term on the right hand side of the inequality is equal to

$$\int_\Omega \det \left( \begin{array}{c} \nabla u_0(x) \\ \nabla U_j^{(2)}(x) \end{array} \right) \, dx,$$

where $u_0$ is the Dirichlet data for the first component of the sequence. Indeed, this assertion is a consequence of the well-known fact that the integral

$$\int_\Omega \det(\nabla V(x)) \, dx$$

only depends on the boundary conditions on $V$. Indeed, in our particular situation, if we call $v_j = U_j^{(1)} - u_0$ we have that

$$\det(\nabla U_j) = \det \left( \begin{array}{c} \nabla v_j + \nabla u_0 \\ \nabla U_j^{(2)} \end{array} \right)$$

$$= \det \left( \begin{array}{c} \nabla u_0 \\ \nabla U_j^{(2)} \end{array} \right) + \det \left( \begin{array}{c} \nabla v_j \\ 0 \end{array} \right) - \operatorname{adj} \left( \begin{array}{c} \nabla u_0 \\ \nabla U_j^{(2)} \end{array} \right)^T \cdot \left( \begin{array}{c} \nabla v_j \\ 0 \end{array} \right).$$
This formula is only valid for $2 \times 2$ matrices. The second term in the right-hand side obviously vanishes. Integrating the equality, we see that the integral of the third term also vanishes, because of the divergence theorem and the fact that the adjoint matrix of a gradient is divergence-free and that the other function involved is the gradient of a function in $H^1_0(\Omega)$. This finishes the proof of the assertion.

By using Young’s inequality we obtain that

$$
\int_{\Omega} \left| \nabla U_j^{(1)}(x) \right|^2 \, dx + \int_{\Omega} \left| \nabla U_j^{(2)}(x) \right|^2 \, dx \leq s \left[ \epsilon \int_{\Omega} \left| \nabla U_j^{(2)}(x) \right|^2 \, dx + \frac{1}{\epsilon} \int_{\Omega} |\nabla u_0|^2 \, dx \right]
$$

for all $j$ and all $\epsilon > 0$. Taking $\epsilon$ small enough we reach the conclusion of the lemma.

**Proof.** (Of Theorem 2). First we prove that $m \geq m$. Let the sequence $\{\nabla U_j\}$ be such that

$$
U_j^{(1)} = u_0 \quad \text{on } \partial \Omega,
\nabla U_j(x) \in \Lambda_x,
$$

for all $j$ and a.e. $x \in \Omega$. Then

$$
\nabla U_j(x) - V^x \in \Lambda = \Lambda_\alpha \cup \Lambda_\beta,
$$

for all $j$ and a.e. $x \in \Omega$. Recall that

$$
\det A = -A^{(1)} \cdot TA^{(2)},
\det A = \alpha \left| A^{(1)} \right|^2 \quad \text{if } A \in \Lambda_\alpha,
\det A = \beta \left| A^{(1)} \right|^2 \quad \text{if } A \in \Lambda_\beta.
$$

Then

$$
\det (\nabla U_j(x) - V^x) = \alpha \left| \nabla U_j^{(1)}(x) \right|^2 \quad \text{if } \nabla U_j(x) - V^x \in \Lambda_\alpha
$$

and

$$
\det (\nabla U_j(x) - V^x) = \beta \left| \nabla U_j^{(1)}(x) \right|^2 \quad \text{if } \nabla U_j(x) - V^x \in \Lambda_\beta.
$$

On the other hand, it is very easy to see that

$$
|\nabla U_j(x) - V^x|^2 = (1 + \alpha) \left| \nabla U_j^{(1)}(x) \right|^2 \quad \text{if } \nabla U_j(x) - V^x \in \Lambda_\alpha
$$

and

$$
|\nabla U_j(x) - V^x|^2 = (1 + \beta) \left| \nabla U_j^{(1)}(x) \right|^2 \quad \text{if } \nabla U_j(x) - V^x \in \Lambda_\beta.
$$

Therefore, we are in hypotheses of Lemma 4 and consequently we obtain that the sequence $\{\nabla U_j\}$ is bounded in $L^2(\Omega; \mathbb{M}^{2\times 2})$. By using Poincare’s inequality, after extracting suitable constants to the second components of the sequence $\{U_j\}$, we have that the sequence $\{U_j\}$ is bounded in $H^1(\Omega; \mathbb{M}^{2\times 2})$. Then if $\nu = \{\nu_x\}_{x \in \Omega}$ is the gradient Young measure associated to $\{\nabla U_j\}$, we have that

$$
\lim_{j \to \infty} \int_{\Omega} W(x, U_j(x), \nabla U_j(x)) \, dx \geq \int_{\Omega} \int_{\mathbb{M}^{2\times 2}} W(x, U(x), A) \, d\nu_x(A) \, dx,
$$

where $U$ is the weak limit of $U_j$ (see [19]).

Let us show the other inequality and the rest of the theorem. This part of the proof was carried out in [4] for the case of a quadratic functional in the gradient of the state. The more general proof given here follows the same ideas as the one in [4], however we have included it for the convenience of the reader. Let $\nu = \{\nu_x\}_{x \in \Omega}$ be a gradient Young measure such that $\text{supp}(\nu_x) \subset \Lambda_x$ as in the statement of the theorem. For the sake of simplicity we assume that $\nu$ is a family of first-order
laminates. By Theorem 1 this assumption is enough for a relaxation result in our situation, although the result remains true for any gradient Young measure as in the statement of the Theorem.

There exists a bounded sequence \( \{ U_j \} \) in \( H^1(\Omega; \mathbb{R}^2) \), \( U_j^{(1)} - u_0 \in H^1_0(\Omega) \), such that the sequence of gradients, \( \{ \nabla U_j \} \), generates \( \nu \) and \( \{ |\nabla U_j|^2 \} \) is equi-integrable ([19]). Therefore, if

\[
\Omega_j = \{ x \in \Omega : \nabla U_j(x) \notin \Lambda_x \},
\]

then \( |\Omega_j| \to 0 \), as \( j \to \infty \) (for being \( \nu_v \) a first order laminate, [19]). By definition of \( \Lambda_x \) there exists a sequence of measurable functions \( E_j : \Omega \to \mathbb{R}^{2 \times 2} \) such that \( E_j(x) \in \{ \alpha I, \beta I \} \) (\( I \) is the identity matrix in \( \mathbb{R}^{2 \times 2} \)), a.e. \( x \in \Omega \), except possibly for \( x \in \Omega_j \). If \( x \in \Omega_j \), \( E_j(x) \) can be chosen such that

\[
E_j(x)\nabla U_j^{(1)} + T \nabla U_j^{(2)} = \nabla p(x).
\]

Hence

\[-\text{div}(E_j\nabla U_j^{(1)}) = P, \quad \text{in } \Omega, \quad U_j^{(1)} = u_0 \quad \text{on } \partial \Omega.\]

We need to use the following very elementary lemma, whose proof can be found in [4].

**Lemma 3.2.** Let \( E_j, \overline{E}_j \to \mathbb{R}^{2 \times 2} \) be such that the problems

\[-\text{div}(E_j\nabla u_j) = P, \quad \text{in } \Omega, \quad u_j = u_0 \quad \text{on } \partial \Omega,\]

\[-\text{div}(\overline{E}_j\nabla \overline{u}_j) = P, \quad \text{in } \Omega, \quad \overline{u}_j = u_0 \quad \text{on } \partial \Omega,\]

admit solution in \( H^1(\Omega) \), \( u_j \) and \( \overline{u}_j \), respectively. Assume that

1. there exists \( \Omega_j \subset \Omega \), \( |\Omega_j| \to 0 \) as \( j \to \infty \), and

\( E_j(x) = \overline{E}_j(x) \quad x \in \Omega \setminus \Omega_j; \)

2. there exists a constant \( C > 0 \), independent of \( j \), such that

\[ ||E_j||_{L^\infty} \leq C \]

for all \( j \);

3. there exists \( \lambda > 0 \), independent of \( j \), such that

\[ E_j(x)v \cdot v \geq \lambda |v|^2 \]

for every vector \( v \in \mathbb{R}^2 \) and a.e \( x \in \Omega; \)

4. the two sequences \( \{ |\nabla u_j|^2 \} \) and \( \{ |E_j\nabla u_j|^2 \} \) are equi-integrable in \( \Omega \).

Then

\[ u_j - \overline{u}_j \to 0, \quad E_j\nabla u_j - \overline{E}_j\nabla \overline{u}_j \to 0 \]

strong in \( H^1(\Omega) \) and \( L^2(\Omega; \mathbb{R}^2) \) resp. In particular \( \{ |\nabla \overline{u}_j|^2 \} \) and \( \{ |\overline{E}_j\nabla \overline{u}_j|^2 \} \) are equi-integrable.

This lemma suggests to consider, for instance, the following sequence of coefficients for the conductivity equation

\[
\overline{E}_j(x) = \begin{cases} E_j(x), & \text{if } x \in \Omega \setminus \Omega_j, \\ \alpha I, & \text{if } x \in \Omega_j. \end{cases}
\]

The definition of \( \overline{E}_j(x) \) for \( x \in \Omega_j \) could be anything as long as \( \overline{E}_j(x) \in \{ \alpha I, \beta I \} \). Associated to this sequence of designs we have the corresponding sequence of pairs of states and stream functions which are the components of the vector field \( \overline{U}_j \)

\[
\overline{E}_j(x)\nabla U_j^{(1)}(x) + T \nabla U_j^{(2)}(x) = \nabla p(x), \quad \text{a.e. } x \in \Omega,
\]
and
\[ \overline{U}^{(1)}_j = u_0, \quad \text{on } \partial \Omega. \]

By construction, \( \nabla U_j \) is in \( A_x \), a.e. \( x \in \Omega \) and all \( j \). We can apply the preceding lemma and conclude that
\[ \nabla U_j - \nabla \overline{U}_j \to 0, \]
strong in \( L^2(\Omega; M^{2 \times 2}) \) and \( \{\nabla U_j\} \) is equi-integrable. In particular, the Young measure generated by \( \{\nabla U_j\} \) is \( \nu \).

On the other hand, by extracting suitable constants to the second components of the terms of \( \{U_j\} \) if necessary, the weak convergence of the gradients yields the weak convergence of \( \{U_j\} \) in \( H^1(\Omega; \mathbb{R}^2) \) to the function \( U \), with
\[ \nabla U(x) = \int_{M^{2 \times 2}} A \, d\nu_x(A). \]

Bearing in mind the bounds (1.3) and that \( W \) is a Carathéodory function whenever is finite we can conclude
\[ \lim_{j \to \infty} \int_\Omega W(x, U_j(x), \nabla U_j(x)) \, dx = \int_\Omega \int_{M^{2 \times 2}} W(x, U(x), A) \, d\nu_x(A) \, dx. \]

As we mentioned before, a consequence of Theorem 2 is that the quasiconvexification of \( W \), \( QW \), is well-defined. Taking into account the admissible functions we are considering in our minimization problem, for a fixed domain \( D \) and \( (x, U) \in D \times \mathbb{R}^2 \), let us consider the envelope
\[ QW(x, U, A) = \inf \left\{ \liminf_{j \to \infty} \frac{1}{|D|} \int_D W(x, U, \nabla V_j(x)) \, dx : \nabla V_j \rightharpoonup A, \; V_j \in H^1(\Omega; \mathbb{R}^2) \right\}. \]

This function is quasiconvex by its own definition. Further, by Theorem 2 we have that
\[ QW(x, U, A) = \min \left\{ \int_{M^{2 \times 2}} W(x, U, F) \, d\nu(F) : \nu \in A, \; \int_{M^{2 \times 2}} F \, d\nu(F) = A \right\}. \]

This implies that the definition in terms of gradients does not depend on the domain and the definition of the quasiconvexification in terms of gradients and in terms of homogeneous gradient Young measures is the same. The resulting integrand is a quasiconvex function.

The proof of Theorem 3 is straightforward using Theorem 2. In particular, bearing in mind the coercivity of the functional \( I \) it is true that
\[ m \geq \tilde{m} \geq \underline{m} \]
(see [19]), and applying Theorem 1 we have
\[ m = \tilde{m} = \underline{m}. \]

The fact that the infimum \( \underline{m} \) is attained is obvious because any minimizing sequence for \( I \) has associated an optimal gradient Young measure for \( \overline{T} \). It is well-known that the first moment of such an optimal gradient Young measure is a minimizer for \( \tilde{m} \), so that this infimum is in fact a minimum. In principle, we do not know whether the functional \( \tilde{I} \) is coercive. However this is true as a consequence of Lemma 4. We will show this in the last section.
4. Examples and final remarks. We would like to discuss some explicit examples where we can apply Theorem 1 and comment on some further remarks. Let us begin by noticing that when either \( g_A(t) \) (in the statement of Theorem 1) is concave or monotone (either increasing or decreasing) over the interval \([r_1(A), r_2(A)]\) the hypothesis on \( g_A(t) \) in Theorem 1 is guaranteed. We also notice that in general we cannot determine at which \( r_i(A) \), \( i = 1, 2 \), the minimum

\[
\min_{i=1,2} \left\{ r_i(A) \varphi_\alpha \left( \frac{1}{r_i(A)} A_1 \right) + \left( 1 - r_i(A) \right) \varphi_\beta \left( \frac{1}{1 - r_i(A)} A_2 \right) \right\}
\]  

(4.8)
is attained. However, it depends on where the absolute minimum of \( g_B(t) \) (in \( 0, 1 \)) lies for any single matrix \( B \). That is to say, if we call \( t_0 \) the point attaining that absolute minimum for a “a-priori” chosen matrix \( B \), then if

\[
t_0 \geq r_2(B)
\]

the minimum in (4.8) is attained in \( r_2(A) \) for all matrices \( A \), and, if

\[
t_0 \leq r_1(B)
\]

the minimum in (4.8) is attained in \( r_1(A) \) for all matrices \( A \). The proof of this fact is an elementary continuity argument.

It is interesting to look at some particular examples where the hypotheses of Theorem 1 hold. For the sake of simplicity, we will take \( \varphi_\alpha = \varphi_\beta = \varphi \) and \( P = 0 \) and focus on the situation where \( \varphi \) only depends on \( \nabla u \) and it is homogeneous of degree \( p \), \( p \geq 0 \). Thus the functions \( g_A(t) \) can be rewritten as

\[
g_A(t) = t^{1-p} \varphi(A_1) + (1-t)^{1-p} \varphi(A_2).
\]

If \( 0 \leq p \leq 1 \) these functions are concave, so that, if \( \varphi \) is convex, we can apply Theorem 1 to determine the value of the quasiconvexification. For example for \( p = 0 \), and overlooking the difficulty appearing because of the singularity at the origen, we get

\[
QW(A) = \begin{cases} 
\varphi(A_2) + r_1(A)(\varphi(A_1) - \varphi(A_2)), & \text{if } A \in \Gamma \text{ and } \varphi(A_1) \geq \varphi(A_2), \\
\varphi(A_2) + r_2(A)(\varphi(A_1) - \varphi(A_2)), & \text{if } A \in \Gamma \text{ and } \varphi(A_1) \leq \varphi(A_2), \\
+\infty, & \text{if } A \notin \Gamma.
\end{cases}
\]

For \( p = 1 \) we curiously obtain

\[
QW(A) = \varphi(A_1) + \varphi(A_2)
\]

if \( A \in \Gamma \). For \( p \geq 1 \) the functions \( g_A(t) \) are now convex, and the further requirement

\[
\frac{\varphi(A_1)^p}{\varphi(A_1)^p + \varphi(A_2)^p} \notin (r_1(A), r_2(A))
\]

must be enforced. When this is the case, having into account a previous remark,

\[
QW(A) = \begin{cases} 
r_1(A)^{1-p} \varphi(A_1) + (1 - r_1(A))^{1-p} \varphi(A_2) & \text{if } \frac{\varphi(A_1)^p}{\varphi(A_1)^p + \varphi(A_2)^p} \leq r_1(A) \\
r_2(A)^{1-p} \varphi(A_1) + (1 - r_2(A))^{1-p} \varphi(A_2) & \text{if } \frac{\varphi(A_1)^p}{\varphi(A_1)^p + \varphi(A_2)^p} \geq r_2(A)
\end{cases}
\]

for \( A \in \Gamma \).

Another interesting example is that in which \( \varphi_\alpha \) and \( \varphi_\beta \) do not depend on \( \nabla u \).

In this case, if we assume for simplicity that \( \varphi_\alpha \leq \varphi_\beta \), is quite easy to realize that

\[
QW(x, U, A) = \begin{cases} 
r_2(A^x) \varphi_\alpha (x, U^{(1)}) + (1 - r_2(A^x)) \varphi_\beta (x, U^{(1)}) & \text{if } A \in A_x, \\
+\infty, & \text{otherwise}
\end{cases}
\]

If there is no dependence on the design \( (\varphi_\alpha \equiv \varphi_\beta) \) \( QW \) does not depend on \( A \).
It is also important to remark that the polyconvexification and the rank-one convexification of $W$ coincide with the quasiconvexification we explicitly compute. Notice that the lower bound for $QW$ is computed by minimizing over Young measures fulfilling equality (2.6), and this is exactly the definition of polyconvexification.

On the other hand, since the optimal microstructures are first-order laminates both convex envelopes coincide with $QW$. In particular, from the practical point of view, this means that there is an optimal design that it locally is a first-order laminated material. This conclusion was also obtained in [23] for the case of the quadratic functional on the gradient of the state.

We mentioned before that the functional $\tilde{I}$ is coercive, in the sense that any sequence verifying the boundary condition on the first component and taking on values where $\tilde{I}$ is finite is bounded in $H^1(\Omega; \mathbb{R}^2)$ after, possibly, extracting suitable constants for the second component of the sequence. In fact, this is again a consequence of Lemma 4. Notice that any matrix $A \in \Gamma$ verifies

$$h(A) \leq 0,$$

i.e.,

$$\alpha \beta \left| A^{(1)} \right|^2 + \left| A^{(2)} \right|^2 \leq (\alpha + \beta) \det A.$$

By the lemma, the set $\Gamma$ is quasiconvex in the sense that any sequence of gradients taking values on $\Gamma$ has a weakly convergent subsequence in $L^2(\Omega; \mathbb{R}^2)$ and the weak limit takes values on $\Gamma$. Arguing as in the proof of Theorem 2 we can prove the coercivity of $\tilde{I}$.

The results shown here provide all the necessary ingredients for the numerical analysis of the optimal design problem. It reduces to find a minimizer of the relaxed problem, whose density we know ($QW$), and then the optimal microstructure is constructed in a standard way [19]. In each point the optimal microstructure can be a first-order laminate, as Theorem 1 shows. In particular, to approximate a minimizer of $\tilde{I}$ we can use finite elements and solve several discrete problems [1].

Another important issue which must be remarked concerns the relationship of the approach shown here with the classical techniques based on G-convergence (in this case, this concept coincides with H-convergence). This was analyzed in [4], and the main conclusion obtained is that a sequence of designs $\{E_j\}$ G-converges to $E$ if and only if

$$\nabla U_j \rightharpoonup \nabla U,$$

weakly in $L^2(\Omega; M^{2 \times 2})$, where

$$E_j(x)\nabla U_j^{(1)}(x) + T\nabla U_j^{(2)}(x) = \nabla f(x)$$

and

$$E(x)\nabla U^{(1)}(x) + T\nabla U^{(2)}(x) = \nabla f(x),$$

a.e. $x \in \Omega$, for all $j$ and all $f \in H^1_0(\Omega)$. This means that we are somehow incorporating the concept of G-convergence in our variational principle because the weak convergence of states and stream functions is just the suitable convergence for weak lower semicontinuity and existence. From this point of view, many questions concerning G-convergence and G-closure can be reinterpreted in terms of the quasiconvexification of $W$, or of the set where $QW$ is finite. In fact, having in mind the above equivalence, it is not hard to prove that

$$\Gamma = \left\{ B \in M^{2 \times 2} : \text{ for some } EB^{(1)} + TB^{(2)} = 0, \ E \in G[\alpha, \beta] \right\}$$
and 

\[ G[\alpha, \beta] = \left\{ E \in \mathbb{M}_{2 \times 2}^2 : \exists B \in \mathbb{M}_{2 \times 2}^2, EB^{(1)} + TB^{(2)} = 0, QW(B) < +\infty \right\}, \]

where \( G[\alpha, \beta] \) is the G-closure of the set of designs \( \{\alpha I, \beta I\} \) (the explicit form of this G-closure can be found, for instance, in [18]). This fact can even be proved doing explicitly the computations, taking into account that we know the inequalities defining \( \Gamma \) and \( G[\alpha, \beta] \). The authors will analyze this issue elsewhere ([2]).

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E-mail address: JoseCarlos.Bellido@uclm.es
E-mail address: Pablo.Pedregal@uclm.es